Problem A.18

The 2×2 matrix representing a rotation of the xy plane is

$$\mathsf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{A.91}$$

Show that (except for certain special angles—what are they?) this matrix has no real eigenvalues. (This reflects the geometrical fact that no vector in the plane is carried into itself under such a rotation; contrast rotations in *three* dimensions.) This matrix *does*, however, have *complex* eigenvalues and eigenvectors. Find them. Construct a matrix S that diagonalizes T. Perform the similarity transformation $\left(\mathsf{STS}^{-1}\right)$ explicitly, and show that it reduces T to diagonal form.

Solution

The aim here is to solve the eigenvalue problem for the given matrix.

$$\mathsf{Ta} = \lambda\mathsf{a}$$

Bring $\lambda\mathsf{a}$ to the left side and combine the terms.

$$(\mathsf{T} - \lambda\mathsf{I})\mathsf{a} = \mathsf{0} \tag{1}$$

Since $\mathsf{a} \neq \mathsf{0}$, the matrix in parentheses must be singular, that is,

$$\det(\mathsf{T} - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\lambda = \cos\theta \pm i\sin\theta.$$

Observe that the eigenvalue λ is real only if θ is an integer multiple of π: $\theta = n\pi$, where $n = 0, \pm1, \pm2, \ldots$. Otherwise, the eigenvalues, $\lambda_+ = \cos\theta + i\sin\theta$ and $\lambda_- = \cos\theta - i\sin\theta$, are complex. To find the corresponding eigenvectors, plug them back into equation (1).

$$(\mathsf{T} - \lambda_+\mathsf{I})\mathsf{a}_+ = \mathsf{0} \qquad\qquad (\mathsf{T} - \lambda_-\mathsf{I})\mathsf{a}_- = \mathsf{0}$$

$$\begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Turn each matrix equation into a system of equations and solve each for either a_1 or a_2.

$$\left.\begin{aligned} -i\sin\theta\, a_1 - \sin\theta\, a_2 &= 0 \\ \sin\theta\, a_1 - i\sin\theta\, a_2 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} i\sin\theta\, a_1 - \sin\theta\, a_2 &= 0 \\ \sin\theta\, a_1 + i\sin\theta\, a_2 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} -ia_1 - a_2 &= 0 \\ a_1 - ia_2 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} ia_1 - a_2 &= 0 \\ a_1 + ia_2 &= 0 \end{aligned}\right\}$$

$$a_2 = -ia_1 \qquad a_1 = -ia_2$$

$$\mathsf{a}_+ = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -ia_1 \end{pmatrix} \qquad \mathsf{a}_- = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -ia_2 \\ a_2 \end{pmatrix}$$

Therefore, the eigenvectors corresponding to $\lambda_+ = \cos\theta + i\sin\theta$ and $\lambda_- = \cos\theta - i\sin\theta$ are

$$\mathsf{a}_+ = a_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \mathsf{a}_- = a_2 \begin{pmatrix} -i \\ 1 \end{pmatrix},$$

respectively, where a_1 and a_2 are arbitrary constants (these are due to the fact that the eigenvalue problem is homogeneous). In order to diagonalize T, let S^{-1} be the 2×2 matrix whose columns are the eigenvectors with $a_1 = a_2 = 1$ for simplicity.

$$\mathsf{S}^{-1} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

Determine S by finding the inverse of this matrix.

$$\begin{aligned} \left(\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ -i & 1 & 0 & 1 \end{array}\right) &\sim \left(\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 2 & i & 1 \end{array}\right) \\ &\sim \left(\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{2} \end{array}\right) \\ &\sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & 1 & \frac{i}{2} & \frac{1}{2} \end{array}\right) \end{aligned}$$

Consequently,

$$\mathsf{S} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Now calculate STS^{-1} and check that the eigenvalues are along the main diagonal with zeros elsewhere.

$$\begin{aligned} \mathsf{STS}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos\theta + i\sin\theta & -i\cos\theta - \sin\theta \\ \sin\theta - i\cos\theta & -i\sin\theta + \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \end{aligned}$$